ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
(ISRA), Impact Factor: 2.114

IJESRT

INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

ZEROS OF A POLYNOMIAL WITH RESTRICTED COEFFICIENTS

M.H. Gulzar
* Department of Mathematics, University of Kashmir, Srinagar 190006, India

ABSTRACT

Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n whose coefficients satisfy $a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0$. Then , a classical result of Enestrom and Kakeya says that all the zeros of P(z) lie in $|z| \le 1$. In this paper , we give some extensions and generalizations of this result.

Mathematics Subject Classification: **30C10, 30C15.**

KEYWORDS and PHRASES: Enestrom-Kakeya Theorem, Polynomial, Zeros.

INTRODUCTION

Regarding the distribution of zeros of a polynomial , Enestrom and Kakeya[10,11] proved the following interesting result:

Theorem A: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0 .$$

Then P(z) has all its zeros in $|z| \le 1$.

In the literature [1-14] there exist several extensions and generalizations of this result. Joyal et al. [9] extended the theorem to polynomials whose coefficients are monotonic but not necessarily non-negative. In fact, they proved the following result:

Theorem B: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 .$$

Then P(z) has all its zeros in

$$|z| \le \frac{1}{|a_n|}(a_n - a_0 + |a_0|) .$$

Govil and Rahman [8] extended the result to the class of polynomials with complex coefficients and proved the following result:

Theorem C: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real α and β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2}, \ j = 0,1,2,.....,n ,$$

and

$$|a_n| \ge |a_{n-1}| \ge ...... \ge |a_1| \ge |a_0| .$$

Then P(z) has all its zeros in the disk

ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
(ISRA), Impact Factor: 2.114

$$|z| \le (\sin\alpha + \cos\alpha) + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}|a_j|.$$

Aziz and Zargar [2] relaxed the hypothesis of Theorem A and proved the following extension of it:

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \ge 1$,

$$ka_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0.$$

Then P(z) has all its zeros in $|z+k-1| \le k$.

Rather and Shah [13] gave the following generalizations of Theorems B ,C and D:

Theorem E: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j, j=0,1,2,......,n.$. If for some reals t,s and $\lambda \in [0,1,2,......,n-1\}$,

$$t+\alpha_n \le \alpha_{n-1} \le ...... \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge ...... \ge \alpha_1 \ge \alpha_0 - s$$

and

$$\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_1 \ge \beta_0 \ge 0,$$

Then all the zeros of P(z) lie in the union of the disks $|z| \le 1$ and

$$\left|z+\frac{t}{a_n}\right| \le \frac{1}{|a_n|}[2\alpha_\lambda - (\alpha_n + t) - \alpha_0 + 2s + |\alpha_0| + \beta_n].$$

Theorem F: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. If for some positive numbers λ and μ ,

$$\lambda + a_n \ge a_{n-1} \ge ...... \ge a_0 - \mu \ge 0,$$

then all the zeros of P(z) lie in the closed disk

$$\left|z+\frac{\lambda}{a_n}\right| \le \frac{1}{a_n}(a_n + \lambda + 2\mu).$$

Gulzar [14] proved the following generalizations of Theorems E and F.

Theorem G: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients and $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j, j=0,1,2,......,n.$ If for some real numbers r , s and $k \ge 1$ and for some $\lambda \in \{0,1,2,......,n-1\}$,

$$r+\alpha_n \le \alpha_{n-1} \le ...... \le \alpha_{\lambda+1} \le k\alpha_\lambda \ge \alpha_{\lambda-1} \ge ...... \ge \alpha_1 \ge \alpha_0 - s$$

and

$$\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_1 \ge \beta_0 \ge 0,$$

then all the zeros of P(z) lie in the union of the disks $|z| \le 1$ and

$$\left|z+\frac{r}{a_n}\right| \le \frac{1}{|a_n|}[2k\alpha_\lambda + 2(k-1)|\alpha_\lambda| - \alpha_n - r - \alpha_0 + 2s + |\alpha_0| + \beta_n].$$

ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
(ISRA), Impact Factor: 2.114

Theorem H: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients and $\text{Re}(a_j)=\alpha_j, \text{Im}(a_j)=\beta_j, j=0,1,2,......,n.$ If for some positive real numbers r , s and $k \geq 1$ and for some $\lambda \in \{0,1,2,......,n-1\}$,

$$r+\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_{\lambda+1} \geq k\alpha_\lambda \geq \alpha_{\lambda-1} \geq ...... \geq \alpha_1 \geq \alpha_0 - s \geq 0$$

and

$$\beta_n \geq \beta_{n-1} \geq ...... \geq \beta_1 \geq \beta_0 \geq 0,$$

then all the zeros of P(z) lie in

$$\left|z+\frac{r}{a_n}\right| \leq \frac{1}{|a_n|}\left[\alpha_n + r + 2(k-1)|\alpha_\lambda| + 2s + \beta_n\right].$$

MAIN RESULTS

In this paper , we prove the following generalizations of Theorems E and F: **Theorem 1:** Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients and $\text{Re}(a_j)=\alpha_j, \text{Im}(a_j)=\beta_j, j=0,1,2,......,n.$ If for some real numbers r , s, u, v and $k \geq 1, l \geq 1$ and for some $\lambda, \mu \in \{0,1,2,......,n-1\}$,

$$r+\alpha_n \leq \alpha_{n-1} \leq ...... \leq \alpha_{\lambda+1} \leq k\alpha_\lambda \geq \alpha_{\lambda-1} \geq ...... \geq \alpha_1 \geq \alpha_0 - s$$

and

$$u+\beta_n \leq \beta_{n-1} \leq ...... \leq \beta_{\mu+1} \leq l\beta_\mu \geq \beta_{\mu-1} \geq ...... \geq \beta_1 \geq \beta_0 - v,$$

then all the zeros of P(z) lie in the union of the disks $|z| \leq 1$ and

$$\left|z+\frac{r+iu}{a_n}\right| \leq \frac{1}{|a_n|}[2(k\alpha_\lambda + l\beta_\mu) + 2(k-1)|\alpha_\lambda| + 2(l-1)\beta_\mu - (\alpha_n+\beta_n) - (r+u)$$
$$-(\alpha_0+\beta_0) + 2s + 2v + |\alpha_0| + |\beta_0|].$$

Taking u=v=0, l=1, $\mu = n, \beta_0 \geq 0$, Theorem 1 reduces to Theorem G.

Theorem 2: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients and $\text{Re}(a_j)=\alpha_j, \text{Im}(a_j)=\beta_j, j=0,1,2,......,n.$ If for some positive real numbers r , s u, v and $k \geq 1, l \geq 1$ and for some $\lambda, \mu \in \{0,1,2,......,n-1\}$,

$$r+\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_{\lambda+1} \geq k\alpha_\lambda \geq \alpha_{\lambda-1} \geq ...... \geq \alpha_1 \geq \alpha_0 - s \geq 0$$

and

$$u\beta_n \geq \beta_{n-1} \geq ...... \geq \beta_{\mu+1} \geq k\beta_\mu \geq \beta_{\mu-1} \geq ...... \geq \beta_1 \geq \beta_0 - v \geq 0,$$

then all the zeros of P(z) lie in

$$\left|z+\frac{r+iu}{a_n}\right| \leq \frac{1}{|a_n|}\left[\alpha_n + \beta_n + r + u + 2(k-1)|\alpha_\lambda| + 2(l-1)|\beta_\mu| + 2s + 2v\right].$$

Taking u=v=0, l=1, $\mu = n, \beta_0 \geq 0$, Theorem 1 reduces to Theorem H.

For different values of the parameters u, v, r, s, k , l , λ, μ in the above results, we get many other interesting results.

 ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
(ISRA), Impact Factor: 2.114

PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+(a_1-a_0z)+a_0\\&=-z^n(a_nz+r+iu)+[(\alpha_n+r-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+......+(\alpha_{\lambda+1}-k\alpha_\lambda)z^{\lambda+1}\\&\quad+(k\alpha_\lambda-\alpha_\lambda)z^{\lambda+1}+(k\alpha_\lambda-\alpha_{\lambda-1})z^\lambda-(k-1)\alpha_\lambda z^\lambda+(\alpha_{\lambda-1}-\alpha_{\lambda-2})z^{\lambda-1}+......\\&\quad+(\alpha_1-\alpha_0+s)z-sz+\alpha_0]+i[(u+\beta_n-\beta_{n-1})z^n+(\beta_{n-1}-\beta_{n-2})z^{n-1}\\&\quad+......+(\beta_{\mu+1}-l\beta_\mu)z^{\mu+1}+(l\beta_\mu-\beta_\mu)z^{\mu+1}+(l\beta_\mu-\beta_{\mu-1})z^\mu\\&\quad+(l\beta_\mu-\beta_\mu)z^\mu+......+(\beta_1-\beta_0+v)z-vz+\beta_0].\end{aligned}$$

For $|z|\ge 1$ so that $\dfrac{1}{|z|^{n-j}}\le 1,\ j=o,1,2,......,n$, we have by using the hypothesis

$$\begin{aligned}|F(z)|&\ge|z|^n|a_nz+r+iu|-[|\alpha_n+r-\alpha_{n-1}||z|^n+|\alpha_{n-1}-\alpha_{n-2}||z|^{n-1}+......+|\alpha_{\lambda+1}-k\alpha_\lambda||z|^{\lambda+1}\\&\quad+(k-1)|\alpha_\lambda||z|^{\lambda+1}+(k-1)|\alpha_\lambda||z|^\lambda+|k\alpha_\lambda-\alpha_{\lambda-1}||z|^\lambda+......+|\alpha_1-(\alpha_0-s)||z|\\&\quad+s|z|+|\alpha_0|+|u+\beta_n-\beta_{n-1}||z|^n+|\beta_{n-1}-\beta_{n-2}||z|^{n-1}+......+|\beta_{\mu+1}-k\beta_\mu||z|^{\mu+1}\\&\quad+(l-1)|\beta_\mu||z|^{\mu+1}+(l-1)|\beta_\mu||z|^\mu+|k\beta_\mu-\beta_{\mu-1}||z|^\mu+......+|\beta_1-\beta_0+v||z|\\&\quad v|z|+|\beta_0|]\\&=|z|^n[|a_nz+r+iu|-\{|\alpha_n+r-\alpha_{n-1}|+\frac{|\alpha_{n-1}-\alpha_{n-2}|}{|z|}+......+\frac{|\alpha_{\lambda+1}-k\alpha_\lambda|}{|z|^{n-\lambda-1}}+\frac{(k-1)|\alpha_\lambda|}{|z|^{n-\lambda-1}}\\&\quad+\frac{(k-1)|\alpha_\lambda|}{|z|^{n-\lambda}}+\frac{|k\alpha_\lambda-\alpha_{\lambda-1}|}{|z|^{n-\lambda}}+......+\frac{|\alpha_1-(\alpha_0-s)|}{|z|^{n-1}}+\frac{s}{|z|^{n-1}}+\frac{|\alpha_0|}{|z|^n}\\&\quad+|u+\beta_n-\beta_{n-1}|+\frac{|\beta_{n-1}-\beta_{n-2}|}{|z|}+......+\frac{|\beta_{\mu+1}-k\beta_\mu|}{|z|^{n-\mu+1}}+\frac{(l-1)|\beta_\mu|}{|z|^{n-\mu+1}}\\&\quad+\frac{(l-1)|\beta_\mu|}{|z|^{n-\mu}}+\frac{|l\beta_\mu-\beta_{\mu-1}|}{|z|^{n-\mu}}+......+\frac{|\beta_1-\beta_0+v|}{|z|^{n-1}}+\frac{v}{|z|^{n-1}}+\frac{|\beta_0|}{|z|^n}\}]\\&\ge|z|^n[|a_nz+r+iu|-\{|\alpha_n+r-\alpha_{n-1}|+|\alpha_{n-1}-\alpha_{n-2}|+......+|\alpha_{\lambda+1}-k\alpha_\lambda|\\&\quad+(k-1)|\alpha_\lambda|+(k-1)|\alpha_\lambda|+|k\alpha_\lambda-\alpha_{\lambda-1}|+......+|\alpha_1-(\alpha_0-s)|+s+|\alpha_0|\\&\quad+|u+\beta_n-\beta_{n-1}|+|\beta_{n-1}-\beta_{n-2}|+......+|\beta_{\mu+1}-k\beta_\mu|+(l-1)|\beta_\mu|\\&\quad+(l-1)|\beta_\mu|+|l\beta_\mu-\beta_{\mu-1}|+......+|\beta_1-\beta_0+v|+v+|\beta_0|\}]\\&=|z|^n[|a_nz+r+iu|-\{\alpha_{n-1}-\alpha_n-r+\alpha_{n-2}-\alpha_{n-1}+......+k\alpha_\lambda-\alpha_{\lambda+1}+2(k-1)|\alpha_\lambda|\end{aligned}$$

ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
(ISRA), Impact Factor: 2.114

$$+k\alpha_\lambda-\alpha_{\lambda-1}+......+\alpha_1-\alpha_0+s+s+|\alpha_0|+u+\beta_n-\beta_{n-1}+\beta_{n-1}-\beta_{n-2}$$
$$+\beta_{\mu+1}-l\beta_\mu+2(l-1)|\beta_\mu|+l\beta_\mu-\beta_{\mu+1}+......+\beta_1-\beta_0+v+v+|\beta_0|\}]$$
$$=|z|^n[|a_nz+r+iu|-\{2(k\alpha_\lambda+l\beta_\mu)+2(k-1)|\alpha_\lambda|+2(l-1)|\beta_\mu|-(\alpha_n+\beta_n)$$
$$-(r+u)-(\alpha_0+\beta_0)+2s+2v+|\alpha_0|+|\beta_n|\}]$$
$$>0$$

if

$$|a_nz+r+iu|>2(k\alpha_\lambda l\beta_\mu)+2(k-1)|\alpha_\lambda|+2(l-1)|\beta_\mu|-(\alpha_n+\beta_n)-(r+u)-(\alpha_0+\beta_0)+2s+2v+|\alpha_0|+|\beta_0|$$

i.e. if

$$\left|z+\frac{r+iu}{a_n}\right|>\frac{1}{|a_n|}[2(k\alpha_\lambda l\beta_\mu)+2(k-1)|\alpha_\lambda|+2(l-1)|\beta_\mu|-(\alpha_n+\beta_n)-(r+u)-(\alpha_0+\beta_0)+2s+2v+|\alpha_0|+|\beta_0|].$$

Thus all the zeros of F(z) whose modulus is greater than or equal to 1 lie in

$$\left|z+\frac{r+iu}{a_n}\right|\le\frac{1}{|a_n|}[2(k\alpha_\lambda l\beta_\mu)+2(k-1)|\alpha_\lambda|+2(l-1)|\beta_\mu|-(\alpha_n+\beta_n)-(r+u)-(\alpha_0+\beta_0)+2s+2v+|\alpha_0|+|\beta_0|].$$

Since the zeros of P(z) are also the zeros of F(z), it follows that all the zeros of F(z) and hence P(z) lie in the union of the disks $|z|\le 1$ and

$$\left|z+\frac{r+iu}{a_n}\right|\le\frac{1}{|a_n|}[2(k\alpha_\lambda l\beta_\mu)+2(k-1)|\alpha_\lambda|+2(l-1)|\beta_\mu|-(\alpha_n+\beta_n)-(r+u)-(\alpha_0+\beta_0)+2s+2v+|\alpha_0|+|\beta_0|].$$

.

This completes the proof of Theorem 1.

Proof of Theorem 2: Consider the polynomial

$$F(z)=(1-z)P(z)$$
$$=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)$$
$$=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}+......+(a_1-a_0z)+a_0$$
$$=-z^n(a_nz+r+iu)+[(\alpha_n+r-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+......+(\alpha_{\lambda+1}-k\alpha_\lambda)z^{\lambda+1}$$

ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
(ISRA), Impact Factor: 2.114

$$+(k\alpha_\lambda-\alpha_\lambda)z^{\lambda+1}+(k\alpha_\lambda-\alpha_{\lambda-1})z^\lambda-(k-1)\alpha_\lambda z^\lambda+(\alpha_{\lambda-1}-\alpha_{\lambda-2})z^{\lambda-1}+......$$
$$+(\alpha_1-\alpha_0+s)z-sz+\alpha_0]+i[(u+\beta_n-\beta_{n-1})z^n+(\beta_{n-1}-\beta_{n-2})z^{n-1}+......$$
$$+(\beta_{\mu+1}-l\beta_\mu)z^{\mu+1}+(l\beta_\mu-\beta_\mu)z^{\mu+1}+(l\beta_\mu-\beta_{\mu-1})z^\mu-(l\beta_\mu-\beta_\mu)z^\mu+......$$
$$+(\beta_1-\beta_0+v)z-vz+\beta_0].$$

For $|z|\geq 1$ so that $\frac{1}{|z|^{n-j}}\leq 1,\ j=o,1,2,......,n$, we have by using the hypothesis

$$|F(z)|\geq |z|^n|a_nz+r+iu|-[|\alpha_n+r-\alpha_{n-1}||z|^n+|\alpha_{n-1}-\alpha_{n-2}||z|^{n-1}+......+|\alpha_{\lambda+1}-k\alpha_\lambda||z|^{\lambda+1}$$
$$+(k-1)|\alpha_\lambda||z|^{\lambda+1}+(k-1)|\alpha_\lambda||z|^\lambda+|k\alpha_\lambda-\alpha_{\lambda-1}||z|^\lambda+......+|\alpha_1-(\alpha_0-s)||z|$$
$$+s|z|+|\alpha_0|+|u+\beta_n-\beta_{n-1}||z|^n+|\beta_{n-1}-\beta_{n-2}||z|^{n-1}+......+|\beta_{\mu+1}-l\beta_\mu||z|^{\mu+1}$$
$$+|l\beta_\mu-\beta_\mu||z|^{\mu+1}+|l\beta_\mu-\beta_{\mu-1}||z|^\mu+(l-1)|\beta_\mu||z|^\mu+......+|\beta_1-\beta_0+v||z|+v|z|+|\beta_0|]$$
$$=|z|^n[|a_nz+r+iu|-\{|\alpha_n+r-\alpha_{n-1}|+\frac{|\alpha_{n-1}-\alpha_{n-2}|}{|z|}+......+\frac{|\alpha_{\lambda+1}-k\alpha_\lambda|}{|z|^{n-\lambda-1}}+\frac{(k-1)|\alpha_\lambda|}{|z|^{n-\lambda-1}}$$
$$+\frac{(k-1)|\alpha_\lambda|}{|z|^{n-\lambda}}+\frac{|k\alpha_\lambda-\alpha_{\lambda-1}|}{|z|^{n-\lambda}}+......+\frac{|\alpha_1-(\alpha_0-s)|}{|z|^{n-1}}+\frac{s}{|z|^{n-1}}+\frac{|\alpha_0|}{|z|^n}$$
$$+|u+\beta_n-\beta_{n-1}|+\frac{|\beta_{n-1}-\beta_{n-2}|}{|z|}+......+\frac{|\beta_{\mu+1}-l\beta_\mu|}{|z|^{n-\mu+1}}+\frac{|l-1||\beta_\mu|}{|z|^{n-\mu+1}}$$
$$+\frac{|l\beta_\mu-\beta_{\mu-1}|}{|z|^{n-\mu}}+\frac{(l-1)|\beta_\mu|}{|z|^{n-\mu}}+......+\frac{|\beta_1-\beta_0+v|}{|z|^{n-1}}+\frac{v}{|z|^{n-1}}+\frac{|\beta_0|}{|z|^n}\}]$$
$$\geq |z|^n[|a_nz+r+iu|-\{|\alpha_n+r-\alpha_{n-1}|+|\alpha_{n-1}-\alpha_{n-2}|+......+|\alpha_{\lambda+1}-k\alpha_\lambda|$$
$$+(k-1)|\alpha_\lambda|+(k-1)|\alpha_\lambda|+|k\alpha_\lambda-\alpha_{\lambda-1}|+......+|\alpha_1-(\alpha_0-s)|+s+|\alpha_0|$$
$$+|u+\beta_n-\beta_{n-1}|+|\beta_{n-1}-\beta_{n-2}|+......+|\beta_{\mu+1}-l\beta_\mu|+(l-1)|\beta_\mu|$$
$$+|l\beta_\mu-\beta_{\mu-1}|+(l-1)|\beta_\mu|+.......+|\beta_1-\beta_0+v|+v+|\beta_0|\}]$$
$$=|z|^n[|a_nz+r|-\{\alpha_n+r-\alpha_{n-1}+\alpha_{n-1}-\alpha_{n-2}+......+\alpha_{\lambda+1}-k\alpha_\lambda$$
$$+2(k-1)|\alpha_\lambda|+k\alpha_\lambda-\alpha_{\lambda-1}+......+\alpha_1-\alpha_0+s+s+|\alpha_0|$$
$$+u+\beta_n-\beta_{n-1}+\beta_{n-1}-\beta_{n-2}+......+\beta_{\mu+1}-l\beta_\mu+2(l-1)|\beta_\mu|$$
$$+l\beta_\mu-\beta_{\mu-1}+......+\beta_1-\beta_0+v+v+|\beta_0|\}]$$
$$=|z|^n[|a_nz+r+iu|-\{\alpha_n+r+2(k-1)|\alpha_\lambda|+2s+\beta_n+u+2(l-1)|\beta_\mu|+2v\}]$$
$$>0$$

if

$$|a_nz+r+iu|>\alpha_n+\beta_n+r+u+2(k-1)|\alpha_\lambda|+2(l-1)|\beta_\mu|+2s+2v$$

ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
(ISRA), Impact Factor: 2.114

i.e. if

$$\left|z+\frac{r+iu}{a_n}\right| > \frac{1}{|a_n|}[\alpha_n+\beta_n+r+u+2(k-1)|\alpha_\lambda|+2(l-1)|\beta_\mu|+2s+2v].$$

Thus all the zeros of F(z) whose modulus is greater than or equal to 1 lie in

$$\left|z+\frac{r+iu}{a_n}\right| \le \frac{1}{|a_n|}[\alpha_n+\beta_n+r+u+2(k-1)|\alpha_\lambda|+2(l-1)|\beta_\mu|+2s+2v].$$

But those zeros of F(z) whose modulus is less than 1 already satisfy the above inequality. In fact, for $|z|\le 1$, we have

$$\begin{aligned}\left|z+\frac{r+iu}{a_n}\right| &\le |z|+\frac{|r+iu|}{|a_n|}\\ &\le 1+\frac{r}{|a_n|}+\frac{u}{|a_n|}\\ &\le \frac{\alpha_n+\beta_n}{|a_n|}+\frac{r}{|a_n|}+\frac{u}{|a_n|}+\frac{2(k-1)|\alpha_\lambda|}{|a_n|}+\frac{2(l-1)|\beta_\mu|}{|a_n|}+\frac{2s}{|a_n|}+\frac{2v}{|a_n|}\\ &= \frac{1}{|a_n|}[\alpha_n+\beta_n+r+u+2(k-1)|\alpha_\lambda|+2(l-1)|\beta_\mu|+2s+2v].\end{aligned}$$

As all the zeros of P(z)are also the zeros of F(z), it follows that all the zeros of F(z) and hence P(z) lie in

$$\left|z+\frac{r+iu}{a_n}\right| \le \frac{1}{|a_n|}[\alpha_n+\beta_n+r+u+2(k-1)|\alpha_\lambda|+2(l-1)|\beta_\mu|+2s+2v].$$

That completes the proof of Theorem 3.

REFERENCES

1. N. Anderson , E.B. Saff and R. S. Verga, An Extension of Enestrom-Kakeya Theorem and its sharpness, SIAM J. Math. Anal. 12(1981),10-22.
2. A. Aziz and B. A. Zargar, some Extensions of of Enestrom-Kakeya Theorem, Glasnik Math. Ser. III 31 (1996), 239-244.
3. A. Aziz and B. A. Zargar, Bounds for the zeros of a Polynomial with Restricted Coefficients, Appl. Math. (Irvine) 3 (2012), 30-33.
4. K. K. Dewan and M. Bidkham, On the Enestrom-Kakeya Theorem, J. Math. Anal. Appl. 180 (1993), 29-36.
5. K. K. Dewan and N. K. Govil, On the Enestrom-Kakeya Theorem, J. Approx. Theory, 42 (1984), 239-243.
6. R. B. Gardner and N. K. Govil, Some Generalizations of the Enestrom-Kakeya Theorem, Acta Math. Hungar. 74 (1997), 125-134.
7. N. K. Govil and G. N. Mctume, Some Extensions of Enestrom-Kakeya Theorem, Int. J. Appl. Math. 11 (2002), 245-253.
8. N. K. Govil and Q. I. Rahman, On the Enestrom-Kakeya Theorem,Tohoku Math. J. 20 (1968), 126-136.
9. M. H. Gulzar, Some Refinements of Enestrom-Kakeya Theorem, Int. JournalmOf Mathematical Archive, Vol. 2 , No. 9, Sep.2011, 1512-1519.
10. A. Joyal, G. Labelle and Q. I. Rahman, On the Location of Zeros of mPolynomials, Canad. Math. Bull. 10 (1967), 55-63.
11. M. Marden, The Goemetry of the Zeros of a Polynomial in a Complex Variable, Math. Surveys No.3, AMS, New York, 1949.
12. G. V. Milovanovic, D. S. Mitrinovic and T. M. Rassias, Topics in Polynomials, Extremal Problems , Inequalities, Zeros, World Scientific Publishing Co., River Edge, NJ, 1994.

ISSN: 2277-9655
Scientific Journal Impact Factor: 3.449
(ISRA), Impact Factor: 2.114

13. N. A. Rather and Mushtaq. A. Shah, On the Location of Zeros of a Polynomial with Restricted coefficients, Acta Et Commentationes universitatis Tartuensis De Mathematica, Vol. 18, No. 2, December 2014, 189-195.
14. M.H.Gulzar, Location of Zeros of a Polynomial with Restricted coefficients, Int. Journal of Advance Foundation and Research in Science and Engineering, Vol. 1, Issue 9, Feb. 2015, 1-9.